Impedance and Polarization-Ratio Transformations by a Graphical Method Using the Isometric Circles*

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Summary—**The isometric circles for the direct and inverse linear fractional transformations can be used for transformations of impedances and polarization ratios. In the loxodromic case an inversion is performed in the isometric circle of the direct transformation, followed by a reflection in the symmetry line of the two circles, and a rotation around the origin of the isometric circle of the inverse transformation. In the nonloxodromic case only the first two operations have to be applied. Three illustrative examples are given: the first shows the transformation of the right half of the complex impedance plane into the unit circle (Smith Chart); the second gives a circular geometric proof of the Weissfloch transformer theorem; the third shows an example of cascading, lossless, two terminal-pair networks.**

Introduction

IN THE SOLUTION of microwave transmission problems, impedance transformations are usually carried out either in the complex impedance (admittance) plane or in the complex reflection coefficient plane. If notations are introduced in accordance with Fig. 1, the input voltage and current are given in terms

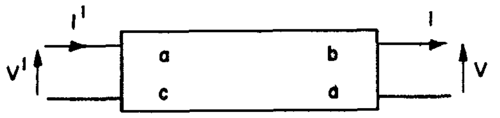

Fig. 1—An arbitrary two terminal-pair network.

of the output voltage and current by the transformation equation

$$\left.\begin{aligned} V' &= aV + bI \\ I' &= cV + dI \end{aligned}\right\}, \tag{1}$$

or in matrix notation by

$$\begin{pmatrix} V' \\ I' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} V \\ I \end{pmatrix}. \tag{2}$$

In a bilateral, two terminal-pair network the determinant of the transformation matrix is identically equal to unity:

$$ad - bc = 1. \tag{3}$$

If we let

$$\left.\begin{aligned} \frac{V'}{I'} &= Z' \\ \frac{V}{I} &= Z \end{aligned}\right\}, \tag{4}$$

we obtain

$$Z' = \frac{aZ + b}{cZ + d}, \qquad ad - bc = 1. \tag{5}$$

This is a linear fractional transformation (also called homographic or bilinear) between the impedances Z and Z'.

The properties of transformation (5) are well known.[1,2] The properties that characterize transformations and lead to their classification are usually the invariants. This transformation (5) which conformally transforms the entire complex plane into itself, is characterized by the invariance of the cross ratio, the fixed points, and the isometric circles. The first two of these have found application in the microwave field but the last one seems to have been completely neglected. In this introductory paper it will be shown that the isometric circles are useful and convenient. The method will be applied to some specific examples.

The Isometric Circles

To find the complete locus of points in the neighborhood of which lengths are unaltered in magnitude by the transformation (5), it is only necessary to study the derivative[2]

$$\frac{dZ'}{dZ} = \frac{1}{(cZ + d)^2}, \qquad ad - bc = 1. \tag{6}$$

The desired locus is clearly the circle

$$|cZ + d| = 1, \qquad c \neq 0 \tag{7}$$

which is called the isometric circle of the direct transformation. Eq. (6) shows that lengths are increased in magnitude within the circle and decreased in magnitude without the circle.

If we solve (5) for Z we get the inverse transformation

$$Z = \frac{-dZ' + b}{cZ - a}, \qquad ad - bc = 1 \tag{8}$$

which has the corresponding isometric circle

$$|cZ - a| = 1. \tag{9}$$

The isometric circle of the direct transformation, C_d, has its center at $O_d = -(d/c)$ and radius $R_c = 1/|c|$; the isometric circle of the inverse transformation has its center at $O_i = a/c$ and the same radius.

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[1] K. Knopp, "Elemente der Funktionentheorie," Sammlung Göschen, Band 1109, Berlin; 1949.

[2] L. R. Ford, "Automorphic Functions," 2nd ed., Chelsea Publishing Co., New York, N.Y.; 1951.

The Relationship Between the Isometric Circles and the Fixed Points of the Transformation

The points that are unchanged by transformation (5), the fixed points, are easily obtained by letting $Z'=Z$, and solving the equation

$$cZ^2 - (a - d)Z - b = 0. \qquad (10)$$

The roots are

$$\zeta_1, \zeta_2 = \frac{a - d \pm \sqrt{(a + d)^2 - 4}}{2c}, \quad ad - bc = 1. \qquad (11)$$

At this point, it is helpful to review the classification of different types of transformation, which can be found in works on the theory of functions.[1,2]

If $a+d$ is real and $|a+d| > 2$, (11) has two real roots. A pure stretching is obtained and the transformation is called *hyperbolic.*

If $a+d$ is real and $|a+d| = 2$, (11) has one root (or two coalescing roots). A pure translation is obtained and the transformation is called *parabolic.*

If $a+d$ is real and $|a+d| < 2$, (11) has two complex conjugate roots. A pure rotation is obtained and the transformation is called *elliptic.*

If, finally, $a+d$ is complex, (11) has two complex roots. A combined stretching and rotation is obtained and the transformation, which can be split into a hyperbolic transformation followed by an elliptic one, or vice versa, is called *loxodromic.*

In accordance with the results of the previous section the distance between the centers of the two isometric circles is $|(a+d)/c|$, while the sum of the two radii is $2/|c|$. Therefore, if we follow the classification given above, we get the hyperbolic case if the two circles are external: the parabolic case, if they are tangent: and the elliptic case, if they intersect. (See Figs. 2–4.) In the loxodromic transformation each circle may have any relation to the other. The positions of the fixed points in relation to the isometric circles can now be obtained easily from (11). The fixed points are marked as crosses in Figs. 2, 3, and 4.

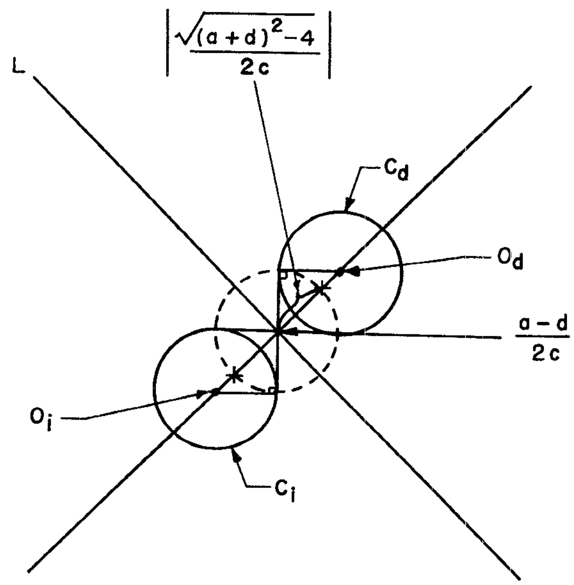

Fig. 2—The hyperbolic case.

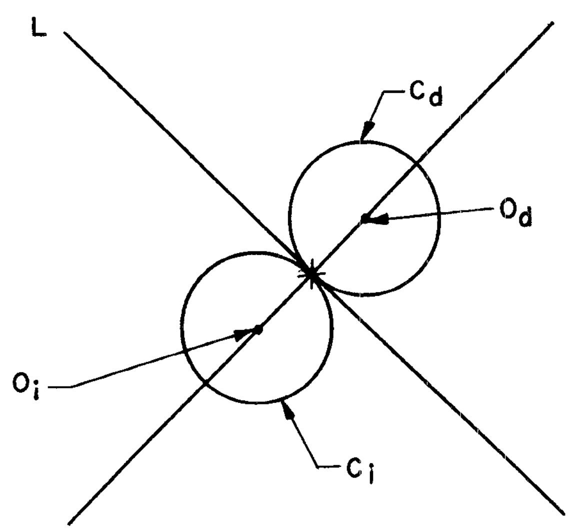

Fig. 3—The parabolic case.

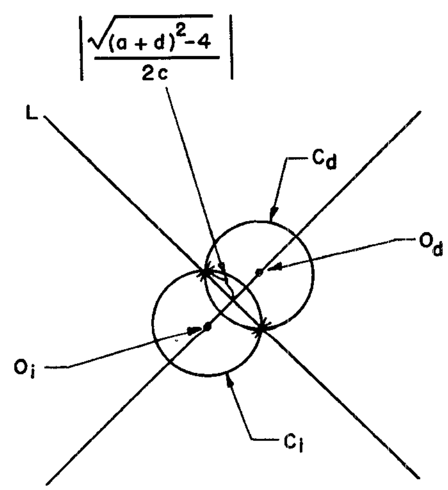

Fig. 4—The elliptic case.

The Graphical Method

In the theory of functions (5) is usually divided in the following way:

$$Z' = \frac{a}{c} - \frac{\frac{1}{c^2}}{Z + \frac{d}{c}}, \qquad ad - bc = 1. \qquad (12)$$

The following operations, suggested by (12), are then usually performed in the complex Z plane (see Fig. 5 on the following page):

1) a translation, $Z_1 = Z + (d/c)$;
2) a complex inversion, $Z_1 Z_2 = 1/|c|^2$;
3) a rotation around the origin through the angle $-2\phi_c$, giving Z_3;

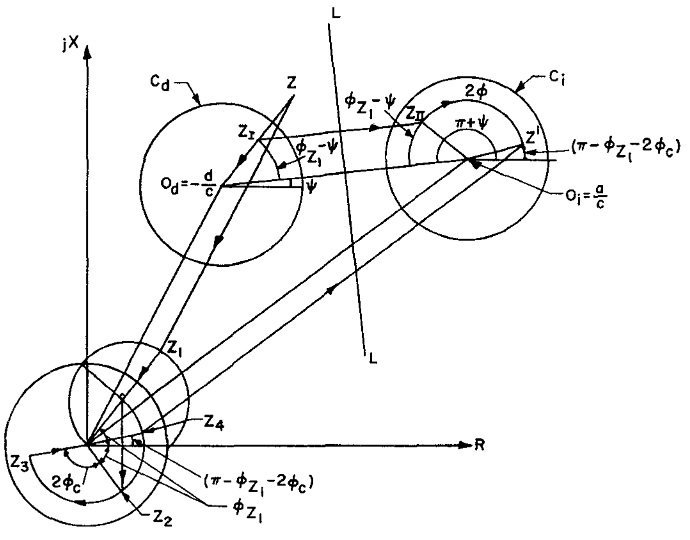

Fig. 5—Graphical interpretation of the linear fractional transformation.

4) a projection through the origin (a rotation around the origin through the angle π), giving Z_4;
5) a translation, $Z'=(a/c)+Z_4$.

If we now draw, as in Fig. 5, the two isometric circles C_d and C_i that have the same radius $R_c=1/|c|$ and the centers $O_d=-(d/c)$ and $O_i=a/c$, the transformation $Z\rightarrow Z'$ can be simplified and the following operations performed: An inversion in the isometric circle C_d of the direct transformation, giving Z_I; a reflection in the symmetry line L of the two circles, giving Z_{II}; and a rotation around the center O_i of the isometric circle of the inverse transformation through an angle -2ϕ.

From Fig. 5 it follows immediately that

$$\psi = \phi - \phi_c \tag{13}$$

or

$$\underline{\left|\frac{a+d}{c}\right.} = \underline{\left|a+d\right.} - \underline{\left|\phi_c\right.}. \tag{14}$$

Thus in the general loxodromic case the transformation (5) can be obtained by an inversion in the isometric circle of the direct transformation, followed by a reflection in the symmetry line L, and, finally, a rotation around the center of the isometric circle of the inverse transformation through an angle -2 arg $(a+d)$. In the nonloxodromic cases, when $a+d$ is real, only the first two operations have to be applied; the same result is obtained by a reflection in L followed by an inversion in the isometric circle of the inverse transformation. This theorem was proved by Ford,[2] but he followed a different line of thought in his proof. He then used the isometric circle to study fundamental regions belonging to linear groups in the theory of automorphic functions.

In network theory graphical inversion methods have been used for symmetric networks, as for example, by König[3] An interesting graphical interpretation of (5) was introduced by Feldtkeller[4] who bases his method on the positions of the so-called oscillation impedances (Schwingwiderstände) which are defined as the roots of the equation

$$-Z = \frac{aZ+b}{cZ+d}. \tag{15}$$

This method, however, is restricted to linear, symmetric networks ($a=d$).

Applications

Transformation of the Right Half of the Complex Impedance Plane in the Unit Circle (Smith Chart)

In order to fulfill the condition in (3), that $ad-bc=1$, the well-known formula for transforming the right half of the complex impedance plane in the unit circle, has to be written in the following way:

$$\Gamma = \frac{\frac{Z}{\sqrt{2}} - \frac{1}{\sqrt{2}}}{\frac{Z}{\sqrt{2}} + \frac{1}{\sqrt{2}}} = \frac{aZ+b}{cZ+d}; \quad ad - bc = 1. \tag{16}$$

Thus, referring to Fig. 6,

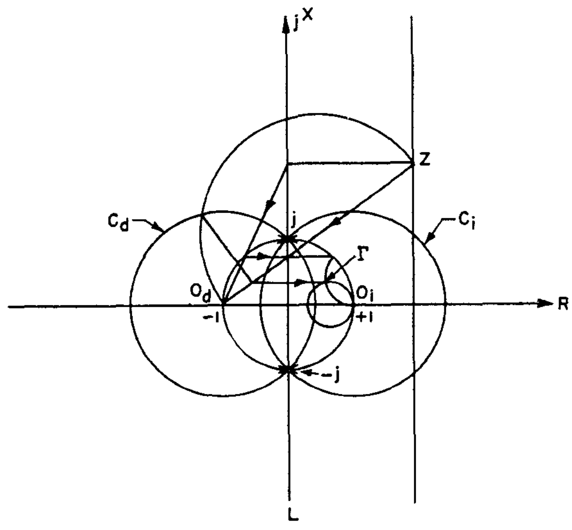

Fig. 6—Transformation of the right half of the complex impedance plane into the unit circle.

$$O_d = -\frac{d}{c} = -1$$
$$O_i = \frac{a}{c} = 1 \tag{17}$$
$$R_c = \sqrt{2}$$
$$a + d = \sqrt{2} = \text{real}.$$

The transformation is clearly nonloxodromic and elliptic, since $a+d$ is real and the isometric circles inter-

[3] H. König, "Über die Abhangigkeit des Scheinwiderstandes eines symmetrischen Vierpols von der Belastung, *Helv. Phys. Acta*, vol. 4, pp. 281–289; 1931.

[4] R. Feldtkeller, "Einführung in die Vierpoltheorie der elektrischen Nachrichtentechnik," S. Hirzel Verlag, Leipzig; 1948.

sect. The fixed points are $\pm j$. Thus Γ is simply obtained from an arbitrary Z by inverting in C_d and reflecting in L, the imaginary axis. The imaginary axis is mapped on the unit circle; the right half plane falls inside the circle. The constant $-R$ and constant $-X$ lines are transformed in two sets of orthogonal circles through the point $+1$. The diagram inside the unit circle is the familiar Smith Chart with Γ defined as the reflection coefficient. An inverse transformation $\Gamma \rightarrow Z$ is simply obtained by inverting in C_i and reflecting in L. If the different planes are stereographically mapped on a Riemann sphere, the transformation constitutes a 90°-rotation around an axis through the fixed points.

The transformation between the Z plane and the Γ plane mentioned above was recently treated by de Buhr.[5] He used the C_i circle as the inversion circle but he did not realize that this circle is one of the isometric circles; nor did he see that his graphical construction is a special case of a more general one.

A New Proof of the Weissfloch Transformer Theorem for Lossless Two Terminal-Pair Networks

Weissfloch's transformer theorem states that for a given frequency any lossless two terminal-pair network can be converted into an ideal transformer by coupling specific lengths of homogeneous transmission lines to each side of the network. The theorem was originally proved by Weissfloch[6] in the complex impedance plane. A proof in the complex reflection coefficient plane was recently given by Weissfloch[7] and by Lueg.[8] It was also proved graphically by Van Slooten,[9] who used the Cayley-Klein diagram (Van Slooten calls it "Cayley-diagram" or "C-diagram"). A new, extremely simple, proof will now be given by means of the isometric circles.

Any impedance transformation through a lossless two terminal-pair network leaves the imaginary axis invariant in the complex plane and the unit circle invariant in the complex reflection coefficient plane. All transformations of this kind that have a common fixed circle the interior of which is transformed into itself, are said to belong to a properly discontinuous group called Fuchsian.[2] The fixed circle mentioned is called the principal circle. One theorem[2] states that the isometric circles of the transformations of a Fuchsian group are orthogonal to the principal circle. In Fig. 7 the lossless network is represented by the isometric circles C_d and C_i, both cutting the principal circle perpendicularly. The network is elliptic, since the circles intersect. The fixed points are marked as crosses. If the circles are separated, the fixed points will coalesce in the tangential case and then continue along the principal circle in the external case. These are the parabolic and the hyperbolic cases. Since the ideal transformer has its fixed points at ± 1, we shall try to find a method for moving the fixed points of Fig. 7[10] to these positions.

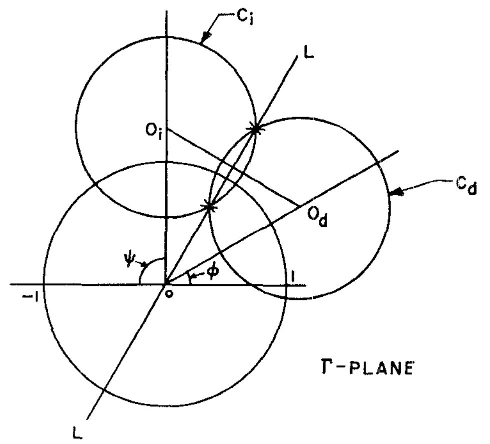

Fig. 7—The isometric circles of an arbitrary, lossless, two terminal-pair network.

A transformation by means of a lossless, homogeneous transmission line constitutes a rotation around the origin in the Γ plane. The isometric circles are straight lines through the origin. With the notations of Fig. 8

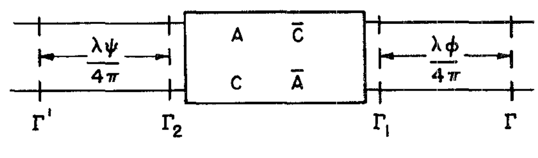

Fig. 8—An arbitrary, lossless, two terminal-pair network between two homogeneous, lossless transmission lines.

we can write

$$\Gamma_1 = e^{-j\phi} \tag{18}$$

$$\Gamma_2 = \frac{A\Gamma_1 + \overline{C}}{C\Gamma_1 + \overline{A}}, \quad |A|^2 - |C|^2 = 1 \tag{19}$$

$$\Gamma' = \Gamma_2 e^{-j\psi} \tag{20}$$

The arbitrary lossless two terminal-pair network has the centers of its isometric circles at $O_d = -(\overline{A}/C)$ and $O_i = A/C$, both having the radius $R_c = 1/|C|$. Eqs. (18) and (19) give

$$\Gamma_2 = \frac{Ae^{-j(\phi/2)}\Gamma + \overline{C}e^{+j(\phi/2)}}{Ce^{-j(\phi/2)}\Gamma + \overline{A}e^{+j(\phi/2)}} \tag{21}$$

having isometric-circle centers at $O_d' = -(\overline{A}/C)e^{j\phi}$ and $O_i' = A/C$ with radii R_c. Thus the isometric circle of the direct transformation is rotated through an angle $+\phi$, while the isometric circle of the inverse transformation is invariant.

[5] J. de Buhr, "Eine neue Methode zur Bearbeitung linearer Vierpole," *FTZ*, pp. 200–204; April, 1955.

[6] A Weissfloch, "Ein Transformationssatz über verlustlose Vierpole und seine Anwendung auf die experimentelle Untersuchung von Dezimeter—und Zentimeterwellen—Schaltungen," *Hochfr. u. Elak.*, vol. 60, pp. 67–73; September, 1942.

[7] A. Weissfloch, "Kreisgeometrische Vierpoltheorie und ihre Bedeutung für Messtechnik und Schaltungstheorie des Dezimeter—und Zentimeterwellengebietes," *Hochfr. u. Elak.*, vol. 61, pp. 100–123; 1943.

[8] H. Lueg, "Über die Transformationseigenschaften verlustloser Vierpole zwischen homogenen Leitungen und ein kreisgeometrischer Beweis des Weissfloch-schen Transformatorsatzes," *AEÜ*, vol. 7, pp. 478–484; 1953.

[9] V. Van Slooten, "Meetkundige Beschouwingen in Verband met de Theorie der Electrische Vierpolen," Thesis, Delft, 1946.

[10] Figs. 7 and 9 should be reflected, so that $\pm 1 \rightarrow \mp 1$.

Eqs. (19) and (20) give

$$\Gamma' = \frac{Ae^{-i(\psi/2)}\Gamma_1 + \overline{C}e^{-i(\psi/2)}}{Ce^{+i(\psi/2)}\Gamma_1 + \overline{A}e^{+i(\psi/2)}} \tag{22}$$

having isometric-circle centers at $O_d'' = -\overline{A}/C$ and $O_i'' = (A/C)e^{-i\psi}$ with radii R_c. Thus the isometric circle of the direct transformation is invariant, while the isometric circle of the inverse transformation is rotated through an angle $-\psi$.

The transmission lines clearly move the fixed points of Fig. 7 to the points ± 1. The lengths of the transmission lines are $\lambda\phi/4\pi$ and $\lambda\psi/4\pi$, where λ is the wavelength. Thus an ideal transformer is obtained and the Weissfloch transformer theorem is proved. If the ideal transformer is represented by

$$Z' = kZ \tag{23}$$

it can easily be proved that the connection between the radius R_c and the impedance transformation ratio k is

$$R_c = \left| \frac{2}{\sqrt{k} - \frac{1}{\sqrt{k}}} \right|. \tag{24}$$

To illustrate this proof given above, a transformation of an arbitrary reflection coefficient Γ, representing, for the sake of simplicity, a reactance, through the ideal transformer is shown in Fig. 9. Here the transformations

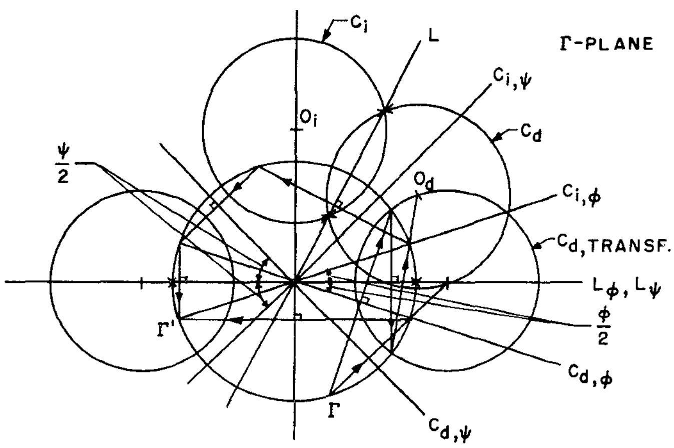

Fig. 9—A circular geometric proof of the Weissfloch transformer theorem using the isometric circles.

performed by the transmission line of length $\lambda\phi/4\pi$, the arbitrary, lossless network, and the transmission line of length $\lambda\psi/4\pi$, are shown to give the same reflection coefficient Γ' as the one obtained by a transformation through the ideal transformer.

Cascading of a Set of Equal Lossless Two Terminal-Pair Networks

In Fig. 10 an arbitrary, lossless, two terminal-pair network is represented by the isometric circles C_d and C_i in the Γ plane. An arbitrary reactance corresponding to the point Γ on the unit circle is transformed in Γ_1. If another network exactly equal to the first one is coupled in series, the new reflection coefficient Γ_2 is obtained at the input by the same operations, inverting in C_d and reflecting in L, as were performed before. For a set of equal networks the reflection coefficients Γ_1, Γ_2, Γ_3, Γ_4, $\cdots$ are obtained. It is seen in Fig. 10 that the transformations correspond to a (non-Euclidean) rotation around the fixed point, which checks with the fact that all transformations are elliptic. Similar constructions can easily be performed in the parabolic and the hyperbolic cases.

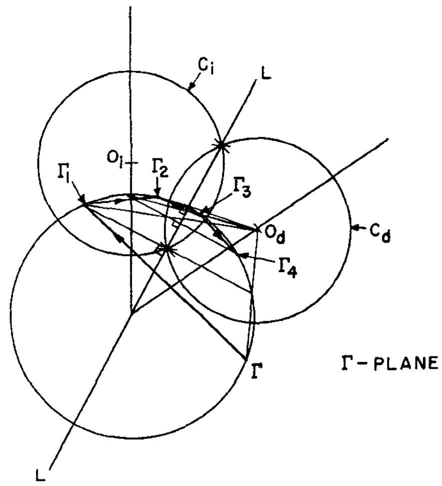

Fig. 10—Transformation through a set of equal, lossless, two terminal-pair networks.

Besides the use of the isometric circle method for impedance transformations, as shown above, the method can also be applied to transformation of polarization ratios.[11]

ACKNOWLEDGMENT

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[11] V. H. Rumsey, G. A. Deschamps, M. L. Kales, and J. I. Bohnert, "Techniques for handling elliptically polarized waves with special reference to antennas," PROC. IRE, vol. 39, pp. 533–552; May, 1951.